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LARGE DEFORMATIONS IN THE THREE-DIMENSIONAL BENDING OF PRISMATIC SOLIDS†

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An accurate non-linear theory of the three-dimensional bending of cylindrical (prismatic) elastic solids is proposed. For three-dimensional bending, each material straight line parallel to the axis of a prismatic beam is converted into a helical line after deformation. All these helical lines have a common axis, orthogonal to the initial axis of the rod. This three-dimensional problem of the non-linear theory of elasticity is reduced to a two-dimensional boundary-value problem for a plane region in the form of a cross-section of the beam. The solution of the two-dimensional problem obtained enables the equilibrium equations to be exactly satisfied in the volume of the cylindrical solid and also enables the boundary conditions to be satisfied on the side surface of the prism. The boundary conditions at the ends of the beam are satisfied in the integral sense. the system of forces acting in the end section of the cylinder is statically equivalent to a force and a moment, which are applied at a point on the axis of the above-mentioned helical lines and directed along this axis. Variational formulations of the non-linear boundary-value problem in a section of the beam subjected to three-dimensional bending are given. © 2004 Elsevier Ltd. All rights reserved.

1. THE TWO-PARAMETER FAMILIES OF FINITE DEFORMATIONS OF A PRISMATIC BEAM

The system of elastostatic equations of an elastic when there are no mass forces consists [1] of the equilibrium equations

$$\operatorname{div}\mathbf{D} = 0 \tag{1.1}$$

the equations of state

$$\mathbf{D} = dW/d\mathbf{C} = \mathbf{P}\cdot\mathbf{C}, \quad \mathbf{P} = 2dW/d\mathbf{G} \tag{1.2}$$

and the geometrical relations

$$\mathbf{G} = \mathbf{C}\cdot\mathbf{C}^T, \quad \mathbf{C} = \operatorname{grad}\mathbf{R}, \quad \mathbf{R} = X_k\mathbf{i}_k \tag{1.3}$$

Here $\mathbf{C}$ is the gradient of the deformation, X_k ($k = 1, 2, 3$) are Cartesian coordinates of particles of the deformed solid (Euler coordinates), $\mathbf{G}$ is the measure of the Cauchy deformation, $\mathbf{i}_k$ are coordinate unit vectors, $\mathbf{D}$ is the asymmetrical Piola stress tensor, $\mathbf{P}$ is the symmetrical Kirchhoff stress tensor, $W(\mathbf{G})$ is the specific potential energy of deformation of an elastic material, and div and grad are the divergence and gradient operators in Lagrange coordinates. We will henceforth use as the latter the Cartesian coordinates of the reference configuration of the solid x_s ($s = 1, 2, 3$). In the case of an isotropic material the specific energy W can be expressed in terms of the invariants of the tensor $\mathbf{G}$

$$I_1 = \operatorname{tr}\mathbf{G}, \quad I_2 = \frac{1}{2}(\operatorname{tr}^2\mathbf{G} - \operatorname{tr}\mathbf{G}^2), \quad I_3 = \det\mathbf{G}$$

System (1.1)–(1.3) can easily be reduced to a system three non-linear scalar equations with unknown functions X_1, X_2 and X_3 and independent variables x_1, x_2 and x_3.

Below we will give particular solutions of system of equations (1.1)–(1.3) containing unknown functions of only two Lagrange coordinates. Each of these solutions represents a two-parameter family of deformations, which are described using the functions

$$X_k = X_k(x_1, x_2, x_3), \quad k = 1, 2, 3$$

We will assume that the elastic solid in the reference configuration has the form of a cylinder (a prism) of arbitrary cross-section. The generating cylinders are parallel to the x_3 axis while the coordinates x_1

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and x_2 are measured in the plane of the cross-section. We will consider the following two-parameter family of deformations of a cylindrical body.

$$\begin{aligned} X_1 &= u_1(x_1, x_2) + lx_3 \\ X_2 &= u_2(x_1, x_2)\cos\omega x_3 - u_3(x_1, x_2)\sin\omega x_3 \\ X_3 &= u_2(x_1, x_2)\sin\omega x_3 + u_3(x_1, x_2)\cos\omega x_3 \\ \omega, l &= \text{const} \end{aligned} \tag{1.4}$$

It is easy to see that, for a deformation of the form (1.4), each material straight line $x_1 = \text{const}$, $x_2 = \text{const}$, parallel to the axis of the cylinder in the reference configuration, after deformation is converted into a simple helical line, the axis of which is the straight line $X_2 = X_3 = 0$. When $l = u_3 = 0$, formulae (1.4) describe the pure bending of a prismatic beam in they x_2x_3 plane investigated previously in [2]. On the basis of relations (1.2)–(1.4) we obtain ($u_{k,\alpha} = \partial u_k/\partial x_\alpha$)

$$\begin{aligned} &\mathbf{C}(x_1, x_2, x_3) = C_{sk}(x_1, x_2)\mathbf{i}_s \otimes \mathbf{j}_k, \quad \mathbf{G} = C_{sm}C_{km}\mathbf{i}_s \otimes \mathbf{i}_k \\ &\mathbf{j}_1 = \mathbf{i}_1, \quad \mathbf{j}_2 = \mathbf{i}_2\cos\omega x_3 + \mathbf{i}_3\sin\omega x_3, \quad \mathbf{j}_3 = -\mathbf{i}_2\sin\omega x_3 + \mathbf{i}_3\cos\omega x_3 \\ &C_{\alpha k} = u_{k,\alpha}, \quad \alpha = 1, 2; \quad k = 1, 2, 3; \quad C_{31} = l, \quad C_{32} = -\omega u_3, \quad C_{33} = \omega u_2 \end{aligned} \tag{1.5}$$

It can be seen that the measure of the Cauchy deformation $\mathbf{G}$ is independent of the x_3 coordinate. If the elastic solid is homogeneous along the x_3 coordinate, the Kirchhoff stress tensor $\mathbf{P}$, by Eq. (1.2), will be a function solely of the coordinates x_1 and x_2. The homogeneity of the body along the x_3 coordinate denotes that the specific elastic energy W can depend explicitly on the x_1 and x_2 coordinates, but does not depend explicitly on x_3: $W = W(\mathbf{G}, x_1, x_2)$. In this case the material may be an anisotropic.

From relations (1.2) and (1.5), for a body that is homogeneous along the x_3 coordinate, we have

$$\mathbf{D}(x_1, x_2, x_3) = D_{sk}(x_1, x_2)\mathbf{i}_s \otimes \mathbf{j}_k \tag{1.6}$$

Taking Eq. (1.6) into account, we can write the equilibrium equations (1.1) in the form

$$D_{11,1} + D_{21,2} = 0, \quad D_{12,1} + D_{22,2} - \omega D_{33} = 0, \quad D_{13,1} + D_{23,2} + \omega D_{32} = 0 \tag{1.7}$$

Taking relations (1.2) and (1.5) into account, we see that Eqs (1.7) represent a system of three scalar equations in three functions of two variables $u_k(x_1, x_2)$ ($k = 1, 2, 3$). If the distribution of the external load $\mathbf{f}$ on the side surface of the prism with unit normal $\mathbf{n} = n_1\mathbf{i}_1 + n_2\mathbf{i}_2$ is given, the boundary conditions on this surface have the form

$$\mathbf{n} \cdot \mathbf{D} = \mathbf{f} \tag{1.8}$$

We will assume that the vector of the distributed load $\mathbf{f}$ can be represented in the form $\mathbf{f} = \mathbf{f}^* \cdot \mathbf{C}$, where the vector $\mathbf{f}^*$ is independent of the x_3 coordinate. Then, the boundary conditions (1.8) for a deformation of the form (1.4) will not contain the variable x_3 and, together with Eqs (1.7), form a two-dimensional boundary-value problem for the plane region in the form of the cross-section of the prism. An example of a surface load, for which the vector $\mathbf{f}^*$ is independent of x_3, is a hydrostatic pressure, uniformly distributed over the side surface.

If $u_3 = l = 0$, then, as shown previously in [2], in the case of an isotropic material the equations $D_{13} = D_{31} = D_{23} = D_{32} = 0$ are satisfied. Hence, the third of Eqs (1.7) is satisfied identically. If, moreover, $\mathbf{f} \cdot \mathbf{j}_3 = 0$, one of the three boundary conditions in (1.8) is also satisfied identically.

In addition to (1.4) there are also families of deformations of a cylindrical beam, for which the initial system of equations of elastostatics can be reduced to a system with two independent variables x_1 and x_2. These families are given by the relations

$$\begin{aligned} X_1 &= \upsilon_1(x_1, x_2)\cos\beta x_3 - \upsilon_3(x_1, x_2)\sin\beta x_3 \\ X_2 &= \upsilon_2(x_1, x_2) + tx_3 \\ X_3 &= \upsilon_1(x_1, x_2)\sin\beta x_3 + \upsilon_3(x_1, x_2)\cos\beta x_3 \\ \beta, t &= \text{const} \end{aligned} \tag{1.9}$$

$$
\begin{aligned}
X_1 &= w_1(x_1, x_2)\cos\psi x_3 - w_2(x_1, x_2)\sin\psi x_3 \\
X_2 &= w_1(x_1, x_2)\sin\psi x_3 + w_2(x_1, x_2)\cos\psi x_3 \\
X_3 &= w_3(x_1, x_2) + \lambda x_3 \\
\psi, \lambda &= \text{const}
\end{aligned}
\tag{1.10}
$$

The family (1.9) is no different in principle from the family (1.4), since it describes the three-dimensional bending of a beam, for which its axis can be converted into a helical line. The axis of this helical line is parallel to the unit vector i_q, whereas for the family (1.4) it is parallel to the unit vector $\mathbf{i}_1$. When $\upsilon_3 = t = 0$, the family (1.9) describes pure bending of a prismatic rod in the x_1x_3 plane.

The family (1.10), represented previously in another form in [3], differs considerably from (1.4) and (1.9), since it describes the axial extension-compression and twisting of a prismatic beam. The formulation and solution of the two-dimensional boundary-value problem on a section of the beam, which arises in the non-linear theory of twisting, can be found in [3, 4].

2. FORMULATION OF THE TWO-DIMENSIONAL BOUNDARY-VALUE PROBLEM ON A CROSS-SECTION OF THE BEAM

We will consider in more detail the formulation of the two-dimensional boundary-value problem for a plane region σ in the form of the cross-section of a prismatic beam, which undergoes three-dimensional bending, i.e. deformation of the form (1.4). We will assume the side surface of the beam to be load-free. The boundary-value problem on the cross-section of the beam consists of boundary conditions (1.8), in which $\mathbf{f} = 0$, and of the equilibrium equations (1.7), in which the quantities D_{sk} are assumed to be expressed in terms of unknown functions of two variables $u_k(x_1, x_2)$ $(k = 1, 2, 3)$ using the governing equations (1.2) and kinematic relations (1.5). The constants ω and l are assumed to be specified parameters.

Suppose $u_k^\circ(x_1, x_2)$ is a certain solution of this boundary-value problem. Then, as can easily be verified, the functions

$$
u_1 = u_1^\circ + L, \quad u_2 = u_2^\circ\cos K - u_3^\circ\sin K, \quad u_3 = u_2^\circ\sin K + u_3^\circ\cos K \tag{2.1}
$$

where K and L are arbitrary real constants, also satisfy Eqs (1.7) and boundary conditions (1.8). This indicates that the position of the elastic body after deformation is determined, apart from rotation around the X_1 axis and translational displacement along the same axis. This non-uniqueness of the solution can be removed by imposing additional conditions on the unknown functions, which eliminate the possibility of arbitrary rotation around the unit vector $\mathbf{i}_1$ and arbitrary displacement along this unit vector. We can use as such conditions, for example, the following integral relations

$$
\iint_\sigma (u_1 - x_1)d\sigma = 0 \tag{2.2}
$$

$$
\iint_\sigma (\cos\theta - 1)d\sigma = 0, \quad \cos\theta = \frac{u_{2,1} + u_{3,2}}{\sqrt{(u_{2,1} + u_{3,2})^2 + (u_{3,1} - u_{2,2})^2}} \tag{2.3}
$$

We can convert boundary-value problem (1.7), (1.8), (2.2), (2.3) on the cross-section of the prism by eliminating the function u_k and taking other quantities as the fundamental unknowns. As a result of eliminating the functions u_k from the expressions for C_{sk} (1.5), we obtain the equations of compatibility of the components of the deformation gradient

$$
C_{11,2} = C_{21,1}, \quad C_{33,\alpha} = \omega C_{\alpha 2}, \quad C_{32,\alpha} = -\omega C_{\alpha 3}; \quad \alpha = 1, 2 \tag{2.4}
$$

It is easy to verify that, in the simply-connected region σ, the functions u_1, u_2 and u_3 are uniquely defined by the specified functions $C_{\alpha k}$, C_{32}, C_{33}, $(\alpha = 1, 2; k = 1, 2, 3)$, which satisfy Eqs (2.4), with the condition that the value of the function u_1 is specified at a certain point of the region σ.

Since, by Eq. (1.2), the components of the Piola stress tensor D_{sk} are expressed in terms of the quantity C_{mn}, the equilibrium equations (1.7), together with the compatibility equations (2.4) form a system of eight equations with eight unknowns. The number of unknown functions $C_{sk}(x_1, x_2)$ $(s, k = 1, 2, 3)$ is equal to eight for the reason that the component C_{31}, according to (1.5), is a known constant. The conditions on the contour $\partial\sigma$ of the cross-section of the beam

$$n_1 D_{1k} + n_2 D_{2k} = 0, \quad k = 1, 2, 3 \tag{2.5}$$

represent non-linear limitations on the values of the functions $C_{sk}(x_1, x_2)$.

Boundary-value problem (1.7), (2.4) and (2.5), which describes the three-dimensional bending of an elastic beam and formulated in terms of the components of the tensor **C**, as can easily be verified, is insensitive to the following replacement of the unknown functions (K = const)

$$\mathbf{C} \to \mathbf{C} \cdot [(\mathbf{E} - \mathbf{i}_1 \otimes \mathbf{i}_1)\cos K + \mathbf{i}_1 \otimes \mathbf{i}_1 - \mathbf{i}_1 \times \mathbf{E}\sin K]$$

where **E** is the unit tensor. This non-uniqueness of the solution can be eliminated using limitations (2.3), in which $\cos\theta$ must now be expressed in terms of C_{sk}

$$\cos\theta = \frac{C_{12} + C_{23}}{\sqrt{(C_{12} + C_{23})^2 + (C_{13} - C_{22})^2}} \tag{2.6}$$

The need for conditions (2.2) obviously disappears if we take the components of the tensor **C** as the unknowns.

Instead of the components of the deformation gradient **C** we can take the components D_{mn} of the Piola stress tensor as the unknown functions. In order to write the compatibility equations (2.4) in terms of the stresses, it is necessary to express the deformation gradient **C** in terms of the tensor **D**. The solution of this problem, in the special case of the pure bending of a beam, when $C_{\alpha 3} = C_{3\alpha} = D_{\alpha 3} = D_{3\alpha} = 0$ $(\alpha = 1, 2)$, has been described previously in [2]. Here we will consider the problem of inverting the relation **D**(**C**) in the more general situation of the three-dimensional bending of a prismatic solid. Assuming the material to be isotropic and following the method previously described in [5], we will first express the positive-definite tensile tensor $\mathbf{U} = (\mathbf{C} \cdot \mathbf{C}^T)^{1/2}$ in terms of the symmetrical Jaumann stress tensor $\mathbf{S} = \mathbf{D} \cdot \mathbf{A}^T$, where $\mathbf{A} = (\mathbf{C} \cdot \mathbf{C}^T)^{-1/2} \cdot \mathbf{C}$ is the orthogonal tensor, which defines the rotations of the material fibres when an elastic solid is deformed. The problem of constructing the relation **C**(**D**) then reduces to representing the rotation tensor **A** in terms of the Piola tensor: $\mathbf{A} = \mathbf{A}(\mathbf{D})$, since the following equalities hold

$$\mathbf{C} = \mathbf{U} \cdot \mathbf{A} = \varphi(\mathbf{S}) \cdot \mathbf{A}(\mathbf{D}) = \varphi[\mathbf{D} \cdot \mathbf{A}^T(\mathbf{D})] \cdot \mathbf{A}(\mathbf{D}) \tag{2.7}$$

Here $\varphi(\mathbf{S})$ is a tensor function, inverse to the function $\mathbf{S} = f(\mathbf{U})$. The relation **A**(**D**) is found as the solution of the equation

$$\mathbf{D} \cdot \mathbf{A}^T = \mathbf{A} \cdot \mathbf{D}^T \tag{2.8}$$

In the problem of the bending of a beam considered here, Eq. (2.8), by (1.5) and (1.6), is equivalent to the following

$$\begin{gathered} \mathbf{D}_0 \cdot \mathbf{A}_0^T = \mathbf{A}_0 \cdot \mathbf{D}_0^T \\ \mathbf{D}_0 = \mathbf{D}|_{x_s = 0} = D_{sk}(x_1, x_2)\mathbf{i}_s \otimes \mathbf{i}_k, \quad \mathbf{A}_0 = \mathbf{A}|_{x_s = 0} \end{gathered} \tag{2.9}$$

The solution of Eq. (2.9) is non-unique and has the form

$$\begin{gathered} \mathbf{A}_0 = \mathbf{K}^{-1} \cdot \mathbf{D}_0 \\ \mathbf{K} = \pm\sqrt{L_1}\mathbf{a}_1 \otimes \mathbf{a}_1 \pm \sqrt{L_2}\mathbf{a}_2 \otimes \mathbf{a}_2 \pm \sqrt{L_3}\mathbf{a}_3 \otimes \mathbf{a}_3 \end{gathered} \tag{2.10}$$

where L_m and $\mathbf{a}_m$ are the eigenvalues and eigen unit vectors of the tensor $\mathbf{D}_0 \cdot \mathbf{D}_0^T$.

Although in sections of the beam x_3 = const, fairly far from the section $x_3 = 0$, the rotations of the material fibres can be extremely large, we can assert that if the parameters ω and l are not too great, the angles of rotation of the material fibres at each point of the section of the beam $x_3 = 0$ will not exceed 90°. Then, as was shown in [5], the unique solution of Eq. (2.9) for $\mathbf{A}_0$ can be separated from the set (2.10) using the inequalities

$$\det \mathbf{A}_0 > 0, \quad \operatorname{tr} \mathbf{A}_0 > 1 \tag{2.11}$$

In the special case of pure bending, relations (2.10) and (2.11) lead [2] to an explicit expression for the tensor $\mathbf{A}_0$ in terms of the tensor $\mathbf{D}_0$. In general, finding the rotation tensor $\mathbf{A}_0$ from formulae (2.10) and (2.11) requires the use of numerical methods.

Thus, taking conditions (2.11) into account we conclude that a unique representation exists of the components of the deformation gradient C_{sk} in terms of the Piola stresses D_{mn}, which enables one to formulate the two-dimensional boundary-value problem (1.7), (2.3) and (2.4)–(2.6) in the stresses D_{mn}.

The next step in transforming the two-dimensional boundary value problem on the cross-sections of the beam consists of satisfying the equilibrium equations (1.7) identically using the stress functions.

We will express the unknown stresses D_{mn} in terms of five new unknowns using the following formulae

$$\begin{aligned} &D_{11} = \Phi_{,2}, \quad D_{21} = -\Phi_{,1}, \quad D_{\alpha\beta} = \omega\Phi_{\alpha\beta}; \quad \alpha = 1,2; \quad \beta = 2,3 \\ &D_{32} = -\Phi_{13,1} - \Phi_{23,2}, \quad D_{33} = \Phi_{12,1} + \Phi_{22,2} \end{aligned} \tag{2.12}$$

Expressions (2.12) identically satisfy the equilibrium equations (1.7) and represent the general solution of these equations. The latter arises from the fact that the functions Φ_{12}, Φ_{22}, Φ_{13}, Φ_{23} are defined uniquely by the stresses specified in the simply-connected region σ, while the function Φ is defined, apart from an arbitrary additive constant, which has no effect on the stressed state of the solid. The quantities Φ, Φ_{12}, Φ_{22}, Φ_{13}, Φ_{23}, called the stress functions, remain subordinate to the compatibility equations (2.4), relations (2.) and (2.6) and boundary conditions (2.5). The latter are written in terms of the stress function as follows (s is the actual length of the arc of the boundary contour):

$$\partial\Phi/\partial s = 0, \quad n_1\Phi_{12} + n_2\Phi_{22} = 0, \quad n_1\Phi_{13} + n_2\Phi_{23} = 0 \tag{2.13}$$

If the cross-section of the beam σ is a multiply connected region, its boundary $\partial\sigma$ consists of the external contour γ_0 and the contours of the apertures γ_t ($t = 1, 2, \ldots, N$). By virtue of the first equality of (2.13) the stress function Φ takes constant values B_0 and B_t on each of the closed curves γ_0 and γ_t. Since the addition of an arbitrary constant to the function Φ has no effect on the stressed state of the beam, without loss of generality we can put $B_0 = 0$. Additional conditions for determining the unknown constants B_t are the integral relations which express the requirement that the function $u_1(x_1, x_2)$ must be unique in the multiply connected region

$$\oint_{\gamma_t} C_{11}dx_1 + C_{21}dx_2 = 0, \quad t = 1, 2, \ldots, N \tag{2.14}$$

The linearity of the boundary conditions (2.13) is an important advantage of the formulation of the boundary-value problem over the cross-section of the beam using the stress functions.

3. THE BOUNDARY CONDITIONS AT THE ENDS OF THE BEAM

The solution of the two-dimensional boundary-value problem on the cross-section of the beam, formulated in Section 2, enables us to satisfy exactly the equilibrium and compatibility equations in the volume of the beam and the boundary conditions on its side surface. This says nothing about the boundary conditions at the end surfaces of the cylinder x_3 = const, which may only be satisfied approximately by appropriate choice of the constants ω and l.

We will determine the principal vector $\mathbf{F}$ and the principal moment $\mathbf{M}$ of the forces acting in an arbitrary cross-section of a cylindrical beam, subjected to deformation of the form (1.4) when there is no load on the side surface. From Eq. (1.6) we have (everywhere henceforth the integration is carried out over the region σ)

$$\begin{aligned} &\mathbf{F}(x_3) = \iint \mathbf{i}_3 \cdot \mathbf{D} d\sigma = F_1\mathbf{i}_1 + F_2\mathbf{j}_2 + F_3\mathbf{j}_3 \\ &F_k = \iint D_{3k} d\sigma, \quad k = 1, 2, 3 \end{aligned} \tag{3.1}$$

where F_k are constant quantities. The necessary equilibrium condition $\mathbf{F}(a) - \mathbf{F}(b)$ of the part of the cylinder bounded by the side surface and the cross-sections $x_3 = a$ and $x_3 = b$, where a and b are arbitrary real numbers, by relations (1.5) and (3.1) leads to the equations

$$s_2F_2 - s_3F_3 = 0, \quad s_3F_2 + s_2F_3 = 0$$
$$s_2 = \cos\omega b - \cos\omega a, \quad s_3 = \sin\omega b - \sin\omega a \tag{3.2}$$

The determinant of system (3.2) with respect to F_2 and F_3 is non-zero, and consequently $F_2 = F_3 = 0$. Hence, the principal vector of the forces in the cross-section of the rod for a deformation of the form (1.4) is the same for all sections x_3 = const and is directed along the X_3 axis. We will now calculate the principal moment $\mathbf{M}$ of the forces in the cross-sections x_3 = const with respect to a certain point on the straight line $X_2 = X_3 = 0$. Since the principal vector is parallel to this straight line, the moment is independent of the choice of the point on the X_1 axis, which enables us to calculate the moment about the point $X_1 = X_2 = X_3 = 0$. Taking into account the fact that $F_2 = F_3 = 0$, using relations (1.4) and (1.5) we obtain

$$\mathbf{M}(x_3) = -\iint[\mathbf{i}_3 \cdot \mathbf{D} \times (u_2\mathbf{j}_2 + u_3\mathbf{j}_3 + u_1\mathbf{i}_1 + lx_3\mathbf{i}_1)]d\sigma = M_1\mathbf{i}_1 + M_2\mathbf{j}_2 + M_3\mathbf{j}_3 \tag{3.3}$$

$$M_1 = \iint(D_{33}u_2 - D_{32}u_3)d\sigma, \quad M_2 = \iint(D_{31}u_3 - D_{33}u_1)d\sigma,$$
$$M_3 = \iint(D_{32}u_1 - D_{31}u_2)d\sigma \tag{3.4}$$

According to expressions (3.4), the quantities M_k ($k = 1, 2, 3$) are constant. From the condition that the moments of all the forces applied to the part of the cylinder between the planes $x_3 = a$ and $x_3 = b$ should balance, we obtain

$$s_2M_2 - s_3M_3 = 0, \quad s_3M_2 + s_2M_3 = 0,$$

whence it follows that $M_2 = M_3 = 0$.

Thus we have proved that the realization of deformation (1.4) requires the application to ends of the cylinder of system of forces which is statically equivalent to the force F_1 and the moment M_1, acting at a point on the axis of a helical line, into which the generatrix of the cylinder is converted after deformation, and directed along this axis. After solving the two-dimensional boundary-value problem in the cross-section, formulated in Section 2, the force and moment become known functions of the parameters ω and l

$$F_1 = F(\omega, l), \quad M_1 = M(\omega, l) \tag{3.5}$$

The conversion of the functions F and M enables the parameters ω and l to be determined for specified values of the force F_1 and the moment M_1. The functions (3.5) possess the following property

$$\partial F/\partial\omega = \partial M/\partial l \tag{3.6}$$

to prove which we will consider the function Π as the linear (i.e. calculated per unit length) potential energy of deformation of an elastic beam, calculated from the solution $u_k(x_1, x_2, \omega, l)$ of the two-dimensional boundary-value problem (1.7), (2.2), (2.3), (2.5)

$$\Pi(\omega, l) = \iint W[u_k(x_1, x_2, \omega, l);\, \omega, l]d\sigma \tag{3.7}$$

Taking into account the formulae

$$\partial\mathbf{j}_2/\partial\omega = x_3\mathbf{j}_3, \quad \partial\mathbf{j}_3/\partial\omega = -x_3\mathbf{j}_2, \quad \partial\mathbf{j}_2/\partial l = \partial\mathbf{j}_3/\partial l = 0$$

and the relations which follow from the symmetry of the Kirchhoff stress tensor

$$C_{ks}D_{kn} = C_{kn}D_{ks}$$

we obtain from relations (1.2), (1.5), (1.6) and (3.7)

$$\frac{\partial\Pi}{\partial\omega} = \iint D_{mn}\frac{\partial C_{mn}}{\partial\omega}d\sigma = \iint(D_{33}u_2 - D_{32}u_3)d\sigma +$$
$$+\iint\left[D_{1k}\frac{\partial}{\partial x_1}\left(\frac{\partial u_k}{\partial\omega}\right) + D_{2k}\frac{\partial}{\partial x_2}\left(\frac{\partial u_k}{\partial\omega}\right) - \omega D_{32}\frac{\partial u_3}{\partial\omega} + \omega D_{33}\frac{\partial u_2}{\partial\omega}\right]d\sigma \tag{3.8}$$

$$\frac{\partial \Pi}{\partial l} = \iint D_{mn}\frac{\partial C_{mn}}{\partial l}d\sigma = \iint D_{31}d\sigma +$$
$$+\iint\left[D_{1k}\frac{\partial}{\partial x_1}\left(\frac{\partial u_k}{\partial l}\right) + D_{2k}\frac{\partial}{\partial x_2}\left(\frac{\partial u_k}{\partial l}\right) - \omega D_{32}\frac{\partial u_3}{\partial l} + \omega D_{33}\frac{\partial u_2}{\partial l}\right]d\sigma \tag{3.9}$$

Integrating by parts and applying Green's formula, it is easy to verify that the second integral on the right-hand side of Eq. (3.8) and the second integral on the right-hand side of Eq. (3.9) vanish by virtue of the equilibrium equations (1.7) and the boundary conditions (2.5). Referring to formulae (3.1) and (3.4), we obtain

$$F_1 = \partial\Pi(\omega, l)/\partial l, \quad M_1 = \partial\Pi(\omega, l)/\partial\omega \tag{3.10}$$

whence relations (3.6) follow.

4. VARIATIONAL FORMULATIONS OF THE TWO-DIMENSIONAL PROBLEM

The non-linear two-dimensional boundary-value problem formulated in Section 2 for a region in the form of the cross-section of the beam allows of different variational formulations, which follow from the variational principles of the non-linear theory of elasticity [6]. Below we derive expressions for some functionals, the stationarity conditions of which are equivalent to the two-dimensional boundary-value problem describing the three-dimensional bending of prismatic solids. Note that these functionals, compared with the similar functionals of the general three-dimensional theory [6], have singularities due to the special form of the deformation of three-dimensional bending (1.4).

The Lagrange-type functional

$$\Pi_1[u_k] = \iint W(u_k)d\sigma \tag{4.1}$$

The functional Π_1 is defined on a set of functions of two variables $u_k(x_1, x_2)$ ($k = 1, 2, 3$), twice differentiable in the region σ, which, according to relations (1.4), specify the field of displacements of the elastic cylinder. The specific deformation energy $W(\mathbf{G})$ is assumed to be expressed in terms of the function u_k using relations (1.5). The condition for the functional (4.1) to be stationary: $\delta\Pi_1 = 0$ is equivalent to the equilibrium equations (1.7) and boundary conditions (2.5), in which the stresses D_{sk} are expressed in terms of the function u_k.

The Hu–Washizu-type functional

$$\begin{aligned} &\Pi_2[C_{\alpha k}, C_{32}, C_{33}, D_{\alpha k}, D_{32}, D_{33}, u_k] = \\ &= \iint[W(C_{\alpha k}, C_{32}, C_{33}) - D_{\alpha k}(C_{\alpha k} - u_{k,\alpha}) - D_{32}(C_{32} + \omega u_3) - D_{33}(C_{33} - \omega u_2)]d\sigma \end{aligned} \tag{4.2}$$

Here and below $\alpha = 1, 2$ and $k = 1, 2, 3$.

In the functional Π_2 the continuously differentiable components of the deformation gradient, the components of the Piola stresses and the function u_k are varied independently. Euler's equations of the variational problem $\delta\Pi_2 = 0$ consist of the equilibrium equations (1.7), the geometrical relations (1.5) and the defining relations of the material in the form

$$D_{\alpha k} = \partial W/\partial C_{\alpha k}, \quad D_{32} = \partial W/\partial C_{32}, \quad D_{33} = \partial W/\partial C_{33} \tag{4.3}$$

Conditions (2.5) serve as the unique boundary conditions for functional (4.2).

The Tonti-type functional

$$\begin{aligned} &\Pi_3[C_{\alpha k}, C_{32}, C_{33}, \Phi, \Phi_{12}, \Phi_{22}, \Phi_{13}, \Phi_{23}, B_1, B_2, \ldots, B_N] = \\ &= \iint[\Phi_{,2}C_{11} + \omega\Phi_{12}C_{12} + \omega\Phi_{13}C_{13} - \Phi_{,1}C_{21} + \omega\Phi_{22}C_{22} + \omega\Phi_{23}C_{23} - \\ &- (\Phi_{13,1} + \Phi_{23,2})C_{32} + (\Phi_{12,1} + \Phi_{22,2})C_{33} - W(C_{\alpha k}, C_{32}, C_{33})]d\sigma \end{aligned} \tag{4.4}$$

Here, the differentiable components of the tensor **C** are comparable with the differentiable stress functions, which satisfy the conditions ($t = 1, 2, \ldots, N$)

$$\Phi|_{\gamma_0} = 0, \quad \Phi|_{\gamma_t} = B_t, \quad n_\alpha \Phi_{\alpha 2}|_{\partial\sigma} = n_\alpha \Phi_{\alpha 3}|_{\partial\sigma} = 0 \tag{4.5}$$

The constants B_t are not specified in advance and are subject to variation. The consequences of the stationarity of the functional Π_3 are the compatibility equations (2.4), which determine relations (4.3), in which the stresses are expressed by formulae (2.12), and the integral relations (2.14).

The Castigliano-type functional

$$\begin{aligned} &\Pi_4[D_{31}, \Phi, \Phi_{12}, \Phi_{22}, \Phi_{13}, \Phi_{23}, B_1, B_2, \ldots, B_N] = \\ &= \iint [V(D_{31}, \Phi, \Phi_{12}, \Phi_{22}, \Phi_{13}, \Phi_{23}) - lD_{31}] d\sigma \end{aligned} \tag{4.6}$$

Here V is the specific additional energy of the elastic material, which is a function of the Piola stress tensor and is connected with the specific potential energy of deformation W by a Legendre transformation

$$V(\mathbf{D}) = \operatorname{tr}[\mathbf{C}^T(\mathbf{D}) \cdot \mathbf{D}] - W(\mathbf{D}), \quad \mathbf{C}(\mathbf{D}) = dV/d\mathbf{D}$$

The method of determining the relation $\mathbf{C}(\mathbf{D})$, necessary to construct the function $V(\mathbf{D})$, is described above in Section 2.

In relation (4.6) we have used representation (2.12) of the Piola tensor in terms of the stress functions, which satisfy the equilibrium equations (1.7) identically. The permissible stress functions must be differentiable and must satisfy conditions (4.5). The conditions for the functional Π_4 to be stationary consist of the compatibility equations (2.4), expressed in terms of the stress functions, the relation $l = \partial V/\partial D_{31}$ and the integral relations (2.14).

The Reissner-type functional

$$\Pi_5[u_k, D_{mn}] = \iint [D_{\alpha k} u_{k,\alpha} + lD_{31} - \omega D_{32} u_3 + \omega D_{33} u_2 - V(D_{mn})] d\sigma \tag{4.7}$$

The functional (4.7) is defined on a set of functions $u_k(x_1, x_2)$, $D_{mn}(x_1, x_2)$, differentiable in the region σ. From the stationarity condition $\delta\Pi_5 = 0$ we obtain the equilibrium equations (1.7), the boundary conditions (2.5) and the defining relations of the elastic material in the following form:

$$u_{k,\alpha} = \partial V/\partial D_{\alpha k}, \quad l = \partial V/\partial D_{31}, \quad \omega u_3 = -\partial V/\partial D_{32}, \quad \omega u_2 = \partial V/\partial D_{33}$$

These variational formulations can be used when solving the non-linear two-dimensional problem over the cross-section of a prism, subjected to three-dimensional bending, using Ritz' method or the method of finite elements.

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